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THE STRUCTURE OF
OPTIMUM INTERPOLATION FUNCTIONS

by

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The form of the approximating function obtained by "optimum interpolation" of meteorological data and related schemes in other disciplines is explored. A variant of Cressman's successive approximation method is shown to be convergent to the same function given by optimum interpolation.

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1. Introduction

The purpose of this report is to review developments in objective analysis of meteorological fields. The first part will review the process known as "optimum interpolation" and observe that it coincides with schemes developed independently in other scientific disciplines. The term "optimum" arises from the fact that the expected mean squared error over some ensemble of realizations (e.g., over time) is minimized. The term "interpolation" is misleading since it refers to inferring values at points other than data points, but not, however, by a scheme that necessarily reproduces given values at the data points. Because of this fact, it would probably be better to call the process "optimum approximation". However, we follow the meteorological literature and retain the term "optimum interpolation".

The second basic thrust of this report is to discuss Cressman's scheme of successive approximations and show that a certain variant of the scheme will converge to the same result given by optimum interpolation. Use of this process could be advantageous from a computational viewpoint, compared to optimum interpolation.

In Section 2 we derive the optimum interpolation scheme and show the functional form of the approximation. We address some computational aspects and recent developments in Section 3. In Section 4 Cressman's successive correction scheme is discussed, including a statistically motivated variant of it. The final section is devoted to showing that a suitable

variant of a successive corrections scheme will converge to the same function as given by optimum interpolation.

2. The Functional Form of Optimum Interpolation

The development of optimum interpolation in meteorology dates back to Gandin [12] and is based on Wiener-Kolmogorov theory in time series analysis. Various disciplines have used similar schemes for some time, apparently developed independently. We have discovered references to developments in geology/mining (where it is usually called kriging) [1], [20], [22], [23], [27], photogrammetry [19], geodesy [15], statistics (where it is called stochastic process prediction) [32], and electrical engineering (where it is sometimes called a Wiener filter) [8].

We derive the general form of optimum interpolation and show the form of the interpolation function. While the latter is known, it is not well known and is even disavowed in print in one paper [reply to 2]. This situation has probably occurred because the principal interest is to obtain a grid of points from scattered observations and not to obtain an approximating surface. However, the form of the equation of the surface is interesting and revealing.

Let $\underline{X}$ be the independent variable, and $Z(\underline{X})$ be a random function whose value is to be estimated from known or measured values $Z(\underline{X}_1)$, $Z(\underline{X}_2)$, ..., $Z(\underline{X}_N)$ at scattered points, $\underline{X}_1$, ..., $\underline{X}_N$. We denote the expected value of $Z(\underline{X})$, $E[Z(\underline{X})]$ by $m(\underline{X})$. This mean value as a function of position is called the trend

surface, and depending on the source of the data, may be assumed to have a particular form, or to be zero. We will assume that the trend surface is given by

$$m(\underline{X}) = \sum_{k=0}^{n} c_k f_k(\underline{X}) , \qquad (n \leq N-1)$$

where the $f_k(\underline{X})$ are known linearly independent functions (with unknown coefficients, c_k). For meteorological applications, Z represents a residual (deviation from climatology, say) which is assumed to have zero mean, and thus $m(\underline{X}) \equiv 0$. We include the term for completeness in our development.

A number of assumptions will be made concerning the distribution of the random function $Z(\underline{X})$. We want to estimate $Z(\underline{X})$ by a linear predictor,

$$\tilde{Z}(\underline{X}) = \sum_{j=1}^{N} \lambda_j Z(\underline{X}_j) .$$

We assume that the optimum predictor is linear in the observed values, which is the case if the distribution is Gaussian, but not necessarily otherwise. In principle the following process can formally be carried out without assumptions about the covariance function

$$(1) \qquad C(\underline{X},\underline{Y}) = E[(Z(\underline{X}) - m(\underline{X}))(Z(\underline{Y}) - m(\underline{Y}))] .$$

In practice, and for computational reasons it is convenient to make the assumptions of stationarity and isotropy for the

covariance function. The net effect of these assumptions is that the covariance function $C(\underline{X},\underline{Y})$ is a function of the distance between $\underline{X}$ and $\underline{Y}$ only, not $\underline{X},\underline{Y}$, or their relative positions other than distance between them. These assumptions probably do not hold in meteorological applications; for example, prevailing winds will certainly tend to give a distortion from isotropy and various landforms will give a distortion from stationarity.

We want to estimate the value of Z at $\underline{X}$; let us call that estimate $\tilde{Z}(\underline{X})$. We will do this by minimizing $E[(Z(\underline{X}) - \tilde{Z}(\underline{X}))^2]$ where

$$\tilde{Z}(\underline{X}) = \sum_{j=1}^{N} \lambda_j Z(\underline{X}_j) ,$$

subject to some conditions which guarantee unbiased estimates. For example if $Z(\underline{X})$ has an unknown constant mean, $E(Z(\underline{X})) = c_o$, then the constraint $\sum_{j=1}^{N} \lambda_j = 1$ is needed to guarantee the estimate is unbiased. In the general case, the constraints to be imposed are

$$(2) \qquad \sum_{j=1}^{N} \lambda_j f_k(\underline{X}_j) = f_k(\underline{X}), \qquad k = 0,1,\ldots,n.$$

Note that this implies we must have $n < N$ and further it can be deduced that if the data lies on the trend surface, so will the estimated point (provided the estimate is unique, a <u>standard assumption</u>).

In meteorological applications it is assumed the measurements ("known" values, $Z(\underline{X}_i)$) are subject to errors, hence the measured values are $Z(\underline{X}_i) + \varepsilon(\underline{X}_i)$. We assume the errors are Gaussian with mean zero, and are independent of the function $Z(\underline{X})$. We denote the covariance function for the errors by $C_\varepsilon(\underline{X},\underline{Y})$. Ultimately we assume the errors are independent, so that we will have $C_\varepsilon(\underline{X},\underline{X}_i) = \sigma^2_{\varepsilon_i}\delta(\underline{X}-\underline{X}_i)$, where $\delta(0) = 1$, $\delta(\underline{X}) = 0$, $\underline{X} \neq 0$. For the derivation, however, we will allow the more general covariance function. We note that in the case of satellite data, for example, the assumption of independence and zero mean will probably not be satisfied.

To minimize $E[(Z(\underline{X}) - \tilde{Z}(\underline{X}))^2]$ subject to the constraints (2), we use Lagrange multipliers, $2\mu_k$, obtaining the objective function

$$(3)\quad E[(Z(\underline{X}) - \sum_{j=1}^{N}\lambda_j(Z(\underline{X}_j) + \varepsilon(\underline{X}_j)))^2] + \sum_{k=0}^{n} 2\mu_k(\sum_{j=1}^{N}\lambda_j f_k(\underline{X}_j) - f_k(\underline{X})).$$

Before differentiation, we write this as

$$(4)\quad E[(Z(\underline{X}) - m(\underline{X}))^2 - 2(Z(\underline{X}) - m(\underline{X}))\sum_{j=1}^{N}\lambda_j(Z(\underline{X}_j) + \varepsilon(\underline{X}_j) - m(\underline{X}_j))$$

$$+ (\sum_{j=1}^{N}\lambda_j(Z(\underline{X}_j) + \varepsilon(\underline{X}_j) - m(\underline{X}_j)))^2]$$

$$+ \sum_{k=0}^{n} 2\mu_k(\sum_{j=1}^{N}\lambda_j f_k(\underline{X}_j) - f_k(\underline{X})).$$

Upon taking partial derivatives with respect to the λ_i and μ_k, and simplifying somewhat, we obtain

$$(5) \quad \sum_j \lambda_j [C(\underline{X}_i,\underline{X}_j) + C_\varepsilon(\underline{X}_i,\underline{X}_j)] + \sum_{k=0}^{n} \mu_k f_k(\underline{X}_i) = C(\underline{X},\underline{X}_i)$$

$$i = 1,2,\ldots,N,$$

and the constraint equations (2).

In matrix form the system of (linear) equations to be solved is conveniently represented in partitioned form as

$$(6) \qquad \left(\begin{array}{c|c} M & F \\ \hline F^t & 0 \end{array}\right) \left(\begin{array}{c} \underline{\lambda} \\ \hline \underline{\mu} \end{array}\right) = \left(\begin{array}{c} \underline{V}_c \\ \hline \underline{V}_f \end{array}\right) \quad ,$$

where

$$M = (C(\underline{X}_i,\underline{X}_j) + C_\varepsilon(\underline{X}_i,\underline{X}_j)) \qquad i,j = 1,\ldots,N$$

$$F = (f_k(\underline{X}_i)) \qquad k = 0,\ldots,n \qquad i = 1,\ldots,N$$

0 is a zero matrix of order $(n+1) \times (n+1)$

$$\underline{\lambda} = (\lambda_1,\lambda_2,\ldots,\lambda_N)^t$$

$$\underline{\mu} = (\mu_0,\ldots,\mu_n)^t$$

$$\underline{V}_c = (C(\underline{X},\underline{X}_i), \ldots, C(\underline{X},\underline{X}_N))^t, \text{ and}$$

$$\underline{V}_f = (f_0(\underline{X}), \ldots, f_n(\underline{X}))^t.$$

Letting G denote the coefficient matrix, we have that (formally) the solution is

$$\begin{pmatrix} \underline{\lambda} \\ --- \\ \underline{\mu} \end{pmatrix} = G^{-1} \begin{pmatrix} \underline{V}_c \\ --- \\ V_f \end{pmatrix}$$

Letting $\underline{d} = (Z(\underline{X}_1) + \varepsilon(\underline{X}_1), \ldots, Z(\underline{X}_N) + \varepsilon(\underline{X}_N), 0, \ldots, 0)$ represent the data vector, we obtain

$$\tilde{Z}(\underline{X}) = \sum_{j=1}^{N} \lambda_j (Z(\underline{X}_j) + \varepsilon(\underline{X}_j)) \tag{7}$$

$$= \underline{d}^t \begin{pmatrix} \underline{\lambda} \\ --- \\ \underline{\mu} \end{pmatrix} = \underline{d}^t G^{-1} \begin{pmatrix} \underline{V}_c \\ --- \\ \underline{V}_f \end{pmatrix} .$$

We will give an alternative interpretation of $\tilde{Z}$. Note that $\begin{pmatrix} \underline{V}_c \\ --- \\ \underline{V}_f \end{pmatrix}$ depends on the value of $\underline{X}$ as well as the data points $\underline{X}_1, \ldots, \underline{X}_N$, while $\underline{d}$ is independent of $\underline{X}$. Since G is symmetric, so is G^{-1}, and we have

$$\tilde{Z}(\underline{X}) = \underline{d}^t G^{-1} \begin{pmatrix} \underline{V}_c \\ --- \\ \underline{V}_f \end{pmatrix} = (G^{-1}\underline{d})^t \begin{pmatrix} \underline{V}_c \\ --- \\ \underline{V}_f \end{pmatrix} .$$

Now, $G^{-1}\underline{d}$ is the solution of a certain system of equations, namely

$$G\underline{a} = \underline{d}, \tag{8}$$

where $\underline{a} = (A_1 \ldots A_N \;\; b_0 \ldots b_n)^t$. This represents $\tilde{Z}(X)$ in the

form

$$(9) \qquad \tilde{Z}(X) = (\underline{d}^t G^{-1}) \begin{pmatrix} \underline{V}_c \\ --- \\ \underline{V}_f \end{pmatrix} = \underline{a}^t \begin{pmatrix} \underline{V}_c \\ --- \\ \underline{V}_f \end{pmatrix}$$

$$= \sum_{i=1}^{N} A_i C(\underline{X},\underline{X}_i) + \sum_{k=0}^{n} b_k f_k(\underline{X}) .$$

The system of equations (8) can be thought of as arising from the requirements that an approximation consisting of a linear combination of the functions $C(\underline{X},\underline{X}_i) + C_\varepsilon(\underline{X},\underline{X}_i)$, $i = 1,\ldots,N$ and $f_k(\underline{X})$, $k = 0,\ldots,n$ be required to interpolate the data, $Z(\underline{X}_i) + \varepsilon(\underline{X}_i)$, $i = 1,\ldots,N$, along with constraints analogous to exactness for the $f_k(\underline{X})$,

$$\sum_{j=1}^{N} A_j f_k(\underline{X}_j) = 0, \quad k = 0,1,\ldots,n .$$

Of course the terms $\sum_{i=1}^{N} A_i C_\varepsilon(\underline{X},\underline{X}_i)$ represent interpolation to the error function and are then dropped to obtain (9). Viewing things from this perspective, the computation of $\tilde{Z}$ at a number of different points is simplified, provided the error estimate given by optimum interpolation is not to be computed. We address this briefly in the next section.

The point of view afforded by (9) makes it apparent that "regionalizing" the process by choosing (from a larger set) data points near the $\underline{X}$ of interest must lead to a discontinuous surface which may, in turn, lead to unnecessary and unwanted disturbances. Phillips [31] addresses this problem when discussing combined analysis and initialization (or perhaps,

better to say, analysis which does not require initialization). See also Williamson and Daley for an iterative approach to overcoming this problem.

In meteorological applications the error covariances are assumed independent [see, e.g., 7], in which case $C_\varepsilon(\underline{X},\underline{X}_i) = \sigma^2_{\varepsilon_i}\delta(\underline{X}-\underline{X}_i)$, where $\sigma^2_{\varepsilon_i}$ is the variance of the error at $\underline{X}_i$. In this instance the matrix M differs from the matrix $(C(\underline{X},\underline{X}_j))$ only in that the diagonal terms are augmented. This has a beneficial effect in terms of the condition number of G and hence the numerical process of solving (6) or (8).

The equations we have derived are for a single dependent variable $Z(\underline{X})$. In both meteorology and geology simultaneous treatment of related dependent variables has been derived and is used. If all dependent variables are measured at each data point, $\underline{X}_i$, and if the sum of the expected squared errors for the dependent estimates is to be minimized, the final result is formally the same as equation (6). However, each entry in M, and each of $Z(\underline{X}_i)$ and λ_j must be vectors, and the entries in the matrix F are replaced by block diagonal matrices. See Myers [24] for details, where the process is called co-kriging.

In meteorology not all variables are measured at each data point. The complication this causes is readily resolved, although it is simpler to group variables instead of points. It is called multivariate optimum interpolation [7], a confusing term to those outside the field since the "multivariate" refers to the dependent variables, not the independent variables. Of course, cross covariances between the dependent variables

are required. See [7] and [21] for the development in meteorological terms.

As a matter of interest, we observe that the process is somewhat reminiscent of Lagrange interpolation, with the λ_j playing the part of the fundamental Lagrange polynomials. Thus, solution of (6) for the λ_j is equivalent to solving for the values of the fundamental Lagrange polynomials at the point $\underline{X}$. Alternatively, solving (8) for the A_j is equivalent to solving for coefficients in the interpolation polynomial expressed as a linear combination of polynomials.

3. Practical Considerations and Recent Results

One of the most important aspects of optimum interpolation is the appropriate specification of the covariance function, $C(\underline{X},\underline{Y})$. In meteorology this has been treated by a number of authors, [3], [9], [33]. The importance of this has been recently noted by Franke [11], Hollingsworth [16], and Lorenc [21] from a practical point of view. In theory, Yakowitz and Szidarovszky [43] have shown that (in the absence of measurement errors) the approximation converges as the set of data points becomes dense. Within some limitations this result holds even if the covariance functions are wrong.

The error estimate for optimum interpolation can be shown (by substitution) to be

$$E[(Z(\underline{X}) - \tilde{Z}(\underline{X}))^2] = C(\underline{X},\underline{X}) - \underline{V}_c^t(2Q^t m^{-1} - Q^t m^{-1} Q^t)\underline{V}_c ,$$

where $Q = I - F(F^t m^{-1} F)^{-1} F^t m^{-1}$. If $E[Z(\underline{X})] = 0$, then $F = 0$ and the above reduces to the more familiar form

$$E[(Z(\underline{X}) - \tilde{Z}(\underline{X}))^2] = C(\underline{X},\underline{X}) - \underline{V}_c^t m^{-1} \underline{V}_c .$$

As noted by Yakowitz and Szidarovsky [43], this estimate is good only if the covariance functions are correct. They show that error estimates with incorrect covariance functions may be so poor as to not converge to zero as the data points become dense, even though the approximation converges. The net result of this is that one should not place too much faith in the error estimates. The covariances assumed are almost certainly wrong, and the more drastic effect is on the error estimate rather than the approximation itself.

Computationally the choice between solving (6), then evaluating $\tilde{Z}(\underline{X})$ by (7), or solving (8), then evaluating $\tilde{Z}(\underline{X})$ by (9) depends on two things: (i) If the error estimate is also to be computed, (6) - (7) is cheaper; (ii) If the error estimate is not to be computed, then (8) - (9) is cheaper, except in the instance of only one evaluation of $\tilde{Z}(\underline{X})$.

In meteorological applications it is impossible to sider all data points at once. This leads to some selection process based on the "most important" observations. Often the closest points are considered the most important. Another scheme is to retain the points corresponding to the larger terms in the covariance matrix $(C(\underline{X}_i,\underline{X}_j))$. Since the importance of a point depends on the entries in the inverse of the matrix rather than the matrix itself, this does not seem to be a good scheme.

A reasonable choice is probably to use "closest points" in some norm which accounts for prevailing influences, e.g., winds.

Several recent papers of practical and theoretical interest relate optimum interpolation to conventional approximation theory. Among these are Kimmeldorf and Wahba [17], Matheron [24], Salkauskas [33], and Wahba [40]. The significance of the result for meteorological applications is discussed by Wahba. For errors which have common variance and covariance functions which have a finite square integral (over the entire plane), optimum interpolation leads to the function $\tilde{Z}(\underline{X})$ given by (7). This function is also the solution of a variational problem in the spatial domain, that is: Find $\underline{Y}(\underline{X})$ (in a certain reproducing kernel Hilbert space) to minimize

$$\sum_{j=1}^{N} (Z(\underline{X}_j) + \varepsilon_j - \underline{Y}(\underline{X}_j))^2 + \lambda J(\underline{Y}) ,$$

where λ is the ratio of the variance of the errors to the variance of the random function $Z(\underline{X})$ $(= C(\underline{X},\underline{X}))$, and $J(\underline{Y})$ is the square norm of $\underline{Y}$ in the Hilbert space. By appropriate choice of the functional $J(\underline{Y})$, new methods can be obtained which minimize or eliminate contributions from unwanted modes.

4. Cressman's Successive Correction Scheme

Cressman's scheme [10] and variations of it [4] are often used for scattered data in meteorology. We will develop the scheme as a matrix iterative process, and show that it may not

converge (although as applied, usually does) if the iteration is continued. In the next section, we will show a relation between a variant of the Cressman scheme and optimum interpolation. We note that Cressman's scheme bears a resemblance to Shepard's method [35], [13], but its differences are more important than its similarities. Most importantly, the gradients of the approximating surface are not necessarily zero at the data points as they are for Shepard's method. In addition, as originally proposed by Cressman, the function is not smooth, i.e., does not have continuous partial derivatives.

Cressman's scheme achieves a weighted average of the data (a convex combination, in fact) as follows. Let the data again be denoted by $Z(\underline{X}_j)$, $j = 1,\ldots,N$, and associate a weight function, $W_j(\underline{X})$, with each point $\underline{X}_j$. This function is ordinarily a univariate function of distance, $||\underline{X} - \underline{X}_j||$. Cressman proposed using

$$(10) \qquad W_j(X) = \begin{cases} \dfrac{D^2 - ||\underline{X} - \underline{X}_j||^2}{||\underline{X} - \underline{X}_j||^2 + D^2}, & ||\underline{X} - \underline{X}_j||^2 < D^2 \\ 0, & ||\underline{X} - \underline{X}_j|| \geq D^2 . \end{cases}$$

Another scheme [4] is to take

$$(11) \qquad W_j(\underline{X}) = \exp(-||\underline{X} - \underline{X}_j||^2/B^2) .$$

A first approximation is taken to be

$$(12) \qquad Z^{(0)}(\underline{X}) = \sum_{j=1}^{N} W_j(\underline{X}) Z(\underline{X}_j) / \sum_{j=1}^{N} W_j(\underline{X}) .$$

The denominator normalizes the weights and since the $W_j(\underline{X})$ are positive, $Z^{(0)}(\underline{X})$ is a convex combination of the data. As such, it satisfies the property $\min_j(Z(\underline{X}_j)) \leq Z^{(0)}(\underline{X}) \leq \max_j(Z(\underline{X}_j))$.

Usually, the process is repeated to correct the differences between the data and the approximation $Z(\underline{X}_j) - Z^{(0)}(\underline{X}_j)$, but using a smaller "radius". This means using a smaller D in (10), or a smaller B in (11). With superscripts denoting iterations, we have the following scheme: Let $Z^{(0)}(\underline{X})$ be given by (12), with $W_j(\underline{X})$ replaced by $W_j^{(0)}(\underline{X})$. Then

$$(13) \quad Z^{(k)}(\underline{X}) = Z^{(k-1)}(\underline{X}) + \sum_{j=1}^{N} W_j^{(k)}(\underline{X})(Z(\underline{X}_j) - Z^{(k-1)}(\underline{X}_j)) / \sum_{j=1}^{N} W_j^{(k)}(\underline{X}), \quad k = 1,2\ldots .$$

If we look at the sequence of vectors which approximate the data vector, $\underline{Z} = (Z(\underline{X}_1)\ldots Z(\underline{X}_N))^t$, we have $\{(Z^{(k)}(\underline{X}_1)\ldots Z^{(k)}(\underline{X}_N))^t\}$, $k = 0,1,\ldots$. Denote these vectors by $\underline{Z}^{(k)}$, and define the matrix

$$(14) \qquad H^{(k)} = \left(\frac{W_j^{(k)}(\underline{X}_i)}{\sum_{p=1}^{N} W_p^{(k)}(\underline{X}_i)} \right), \quad i,j = 1,\ldots,N.$$

Then the iteration takes the form

$$(15) \qquad \begin{aligned} \underline{Z}^{(0)} &= H^{(0)}\underline{Z} \\ \underline{Z}^{(k)} &= \underline{Z}^{(k-1)} + H^{(k)}(\underline{Z} - \underline{Z}^{(k-1)}), \quad k = 1,2,\ldots \end{aligned}$$

Formally, the latter can be written as

$$\underline{Z}^{(k)} = H^{(k)}\underline{Z} + (I - H^{(k)})\underline{Z}^{(k-1)} ,$$

and thus

$$\underline{Z} - \underline{Z}^{(k)} = (I - H^{(k)})(\underline{Z} - \underline{Z}^{(k-1)}), \quad k = 1,2,...$$

This easily leads to

$$\underline{Z} - \underline{Z}^{(k)} = [\prod_{p=1}^{k} (I - H^{(p)})](\underline{Z} - \underline{Z}^{(0)}) , \tag{16}$$

and we see that this iteration converges provided that, for sufficiently large k, the norms of the $I-H^{(k)}$ are bounded by a constant less than one. This holds for any norm; hence, if all eigenvalues of the $I - H^{(k)}$ have magnitude bounded by a constant less than one, for sufficiently large k, convergence is obtained.

Generally, the effect of decreasing D in (10) or B in (11) is to increase the relative size of the diagonal elements in $H^{(k)}$. Since each row sums to one, the matrix will eventually become diagonally dominant. In any case, if all eigenvalues are positive (as for (11), for example), the eigenvalues of $I - H^{(k)}$ are then bounded by a constant less than one, independent of k, provided B is a decreasing function of k.

The situation for weights given by (10) is not so pleasant. In this case, the matrix $H^{(k)}$ may have negative eigenvalues, which leads to $I - H^{(k)}$ having eigenvalues greater than one.

As D decreases, all eigenvalues do become positive. Whether or not this happens for values of D used in practice should be investigated since this has a bearing on the stability of the iteration.

The effect of decreasing D in (10) or B in (11) is to speed convergence of the iteration, since this tends to increase the eigenvalues of $H^{(k)}$. We note in passing that $H^{(k)}$ is a stochastic matrix, and thus has its largest eigenvalue equal to one, with eigenvector $(1,\ldots 1)^T$. In terms of the approximation, this means convergence in one iteration if $\underline{Z}$ is a constant vector, i.e., $\underline{Z} = (c,\ldots,c)^t$, c is a constant. The maximum/minimum principle cited earlier implies the scheme is exact for constants however.

In the next section we discuss the general form of the approximation, and show that under certain simple modifications the scheme approximates optimum interpolation.

5. Relation of a Variant of Cressman's Scheme to Optimum Interpolation

The general form of $Z^{(0)}(\underline{X})$ in (12) is a rational function in the weights, $W_j(\underline{X})$, and if the scheme is iterated as in (13), the form is that of a sum of functions, each rational in the appropriate set of weights $W_j^{(k)}$, $k = 0,1,\ldots$. If weights were taken to be the functions $C(\underline{X},\underline{X}_j) + C_\varepsilon(\underline{X},\underline{X}_j)$ in (1), for all k, the resulting approximation bears some relationship to optimum interpolation, although it is rational in the covariance functions rather than linear in them. However, if the denominators of (12) and (13) (and hence, of $H^{(k)}$) are replaced

by a suitable constant, the iteration will converge to the optimum interpolation function.

First we consider the covariance matrix

$$(17)\qquad M = (C(\underline{X}_i,\underline{X}_j) + C_\varepsilon(\underline{X}_i,\underline{X}_j)), \quad i,j = 1,\ldots,N .$$

This matrix must be positive semidefinite and we make the usual assumption that it is definite, i.e., has no zero eigenvalues. Let $||M||$ denote the max row sum norm of M, and let β be a constant satisfying $\beta > \frac{1}{2}||M||$. Now consider the iteration obtained by replacing the denominators in (12) and (13) by β, which leads to the matrix iteration analogous to (15)

$$(18)\qquad \begin{aligned} \underline{Z}^{(0)} &= \tfrac{1}{\beta}M\underline{Z} \\ \underline{Z}^{(k)} &= \underline{Z}^{(k-1)} + \tfrac{1}{\beta}M(\underline{Z} - \underline{Z}^{(k-1)}), \quad k = 1,2,\ldots . \end{aligned}$$

This leads to the analogue of (16),

$$(19)\qquad \underline{Z} - \underline{Z}^{(k)} = (I - \tfrac{1}{\beta}M)^k(\underline{Z} - Z^{(0)}) .$$

Thus, convergence is obtained whenever $I - \frac{1}{\beta}M$ has all eigenvalues strictly bounded by one. Since the eigenvalues of $\frac{1}{\beta}M$ must be bounded by $||\frac{1}{\beta}M|| < 2$ by our choice of β, convergence is obtained. The form of each $Z^{(k)}(\underline{X})$ is a linear combination of the $C(\underline{X},\underline{X}_j) + C_\varepsilon(\underline{X},\underline{X}_j)$. Because convergence implies agreement at the data points, we see that if the error covariances

have the form noted before, $C_\varepsilon(\underline{X},\underline{X}_i) = \sigma^2_{\varepsilon_i}\delta(\underline{X}-\underline{X}_i)$, the limiting approximation given by (18) agrees exactly with that given by (8) - (9), except at the $\underline{X}_i$, where a jump occurs to yield "interpolation." Of course one thinks in terms of dropping that term for the final approximation, but must do so only if an evaluation point coincides with a grid point. Practically speaking, the reverse situation is where the special instance is encountered.

The rate of convergence may be slow because of the likelihood of M having small eigenvalues, leading to $I - \frac{1}{\beta}M$ having eigenvalues close to one. However, the presence of the error covariance tends to increase the eigenvalues of M, and in this respect, large observational errors would benefit the convergence rate. It would seem best to try to choose β to minimize the magnitude of the eigenvalue of $I - \frac{1}{\beta}M$ of largest magnitude. This would maximize the rate of convergence. On the other hand, most of the significant information may correspond to large eigenvalues of M. (Recall that one is an eigenvalue of $H^{(k)}$ with eigenvector $(1,1,\ldots,1)^t$.) In this case, it would make sense to take $\beta \simeq ||M||$ which would cause rapid convergence for these modes, while modes corresponding to small eigenvalues are of high frequency and could be best filtered out.

The filtering potential of this scheme should be investigated further to determine whether or not the eigenvectors corresponding to small eigenvalues do indeed lead to unwanted noise in the approximation which later must be filtered out.

If so, this scheme could be an advantageous one to use. Some simulations of the scheme have been carried out through the iteration process. However, the results are nontrivial to interpret and need additional study, particularly in the light of the filtering scheme presently used in the operational model. The combination of including the measurement errors and the constant normalization factor will result in the successive correction method appearing more like optimum interpolation. A multivariate scheme could be derived in a straightforward fashion.

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